ADVANCES IN MATHEMATICS **45**, 73–91 (1982)

Degeneracy Indices and Chern Classes*

KUO–TSAI CHEN

Department of Mathematics, University of Illinois, at Urbana-Champaign, Urbana, Illinois 61801

Let E be a complex k-plane bundle over a compact oriented manifold N. The degeneracy set Δ of sections $s_1,..., s_r$ consists of all points y of N, at which $s_1 y,..., s_r y$ are linearly dependent. In the holomorphic case, it is known that the degeneracy set can be interpreted as to represent the Poincaré dual of the k'th Chern class, $k' = k - r + 1$. This result can be traced back to Chern. A complete proof is given in Cornalba and Griffiths [4] using complex analytic techniques. (See also an earlier proof by Griffiths [5] under more restrictive conditions.) In Griffiths and Harris [6], there is a C^∞ version of the result under some transversality requirements.

Since holomorphic sections can be scarce, it is very much desirable to have such a C^∞ version. The result becomes applicable to Pontryagin classes. In this work, we treat the C^∞ case by attaching multiplicities to the degeneracy set and prove a C^∞ version of the result in generality (Theorem 4.1). In the special case of $r = 1$ and $\dim N = 2k$, the theorem reduces to the Hopf index theorem for complex vector bundles. Disregarding technical details, Theorem 4.1 improves the theorem of Griffiths and Harris by the removal of transversality conditions. Restricted to the holomorphic case, our result contributes an explicit formulation of multiplicities in terms of indices of Hopf type. We also obtain a generalization of the classical Bezout theorem not only for polynomials but also for matrices with polynomial entries.

We need a notion, which will cover both complex and real analytic varieties as well as smooth manifolds. For this purpose the notion of stratified C^∞ sets is given in Section 1. Lemma 2.1 expresses intersection numbers through a residue formula. (The notion of residue here is somewhat different from that given by King [7].) Theorem 3.1 is an essential preliminary result dealing with Poincaré duality under a C^∞ map. In Section 4, the degeneracy multiplicity is formulated. We use the sections $s_1,..., s_r$ to construct a C^∞ map f from the base manifold N to a complex Grassmannian so that Theorem 3.1 becomes available.

* Work supported in part by NSF MCS 79–00321.

0001-8708/82/070073–19$05.00/0

The proof of Theorem 4.1 is divided into two parts. In the first part, the degeneracy set Δ is shown to be precisely the preimage of a Schubert variety dual to the k'th Chern class of the canonical k-plane bundle of the mentioned complex Grassmannian under map f. Theorems 3.1 and 4.1 involve respectively two different notions of multiplicities. The second part of the proof, namely Section 5, is devoted exclusively to the verification of their agreement.

In Section 6, we state complex and real analytic versions of the main result. A brief discussion on Pontryagin classes is done in Section 7.

As an application, we state, in Section 8, a generalized Bezout theorem on loci of rank deficiency of an $r \times k$ polynomial matrix $f(z, \bar{z}) = (f_{ij}(z, \bar{z}))$, $r \leqslant k$, where each entry $f_{ij}(z, \bar{z})$ is a polynomial homogeneous of degree d_j' in the complex variables $z = (z_0,..., z_n)$ and of degree d_j'' in the conjugate variables $\bar{z}$. Loci of rank deficiency, with multiplicities counted, are taken as cycles, whose homology classes in CP^n can be determined in terms of the integers $d_j' - d_j''$, $j = 0,..., n$.

A smooth version of Theorem 4.1 with $r = 1$ has been announced in [2]. A restricted version of Theorem 3.1 can be found in [1].

At this point, we would like to explain some conventions used in the text of this work: Let N be a C^∞ manifold equipped with a Riemann metric. Let Γ be a submanifold of N. By the ε-tubular neighborhood $T(\varepsilon)$ of Γ in N, we mean the image of the associated ε-ball bundle of the normal bundle of Γ under the geodesic exponential map. Thus, $T(\varepsilon)$, if well-defined, is a fiber bundle over Γ with a projection map $\pi: T(\varepsilon) \to \Gamma$. If K is a subset of Γ, we shall abuse the term "neighborhood" and call $\pi^{-1}K$ the ε-tubular neighborhood of K.

Every C^∞ homotopy $N \times I \to N$ between two C^∞ maps $g_0, g_1: N \to N$ induces a cochain homotopy $\int: \Omega(N) \to \Omega(N)$ of the de Rham complex $\Omega(N)$ such that

$$d\int + \int d = g_1^* - g_0^*.$$

For every p-form u on N, the $(p-1)$-form $\int u$ is obtained from u through an integration with respect to the "time" parameter. This is the basic idea behind the Poincaré lemma.

Every well-defined ε-tubular neighborhood $T(\varepsilon)$ of Γ has an obvious C^∞ homotopy $T(\varepsilon) \times I \to T(\varepsilon)$ between the composite map $i\pi$ and the identity map, where $i: \Gamma \to T(\varepsilon)$ is the inclusion. Let $\int$ be the resulting cochain homotopy of the de Rham complex of $T(\varepsilon)$. Then, for every closed p-form τ on $T(\varepsilon)$, we have

$$d\theta = \tau - \pi^*(\tau | \Gamma),$$

where $\theta = \int \tau$. The homotopy deforms every point y of $T(\varepsilon)$ to the point πy along a geodesic. If τ vanishing along this geodesic, so does θ. We shall say that θ is obtained by using the Poincaré lemma based on the deformation retraction of the tubular neighborhood $T(\varepsilon)$.

1. Stratifications

In this section, the notion of stratified C^∞ sets is introduced in order to include both C^∞ submanifolds and analytic subvarieties.

Definition. Let L be a closed subset of a C^∞ manifold M. A C^∞ stratification of L is a descending sequence of closed subsets

$$L = L_r \supset \cdots \supset L_0 \supset L_{-1} = \varnothing \tag{1.1}$$

having the following properties:

(a) About each point of $L_i - L_{i-1}$, $0 \leqslant i \leqslant r$, L_i is locally an i-dimensional C^∞ submanifold of M.

(b) Each $L_i - L_{i-1}$ has only a finite number of connected components.

If L has a C^∞ stratification as above with $L_r - L_{r-1}$ nonempty, then L is said to be an r-dimensional stratified C^∞ set in M. We say that L is oriented if the manifold $L_r - L_{r-1}$ is oriented.

Lemma 1.1. *Let L be a compact r-dimensional stratified C^∞ set in a C^∞ manifold M. If τ is a closed p-form on M with $p \geqslant r$, then there exists a closed p-form τ' on M having the following properties:*

(a) *$\tau - \tau'$ is exact on M.*

(b) *τ' vanishes about the closure of a neighborhood U of L_{r-1} in M.*

(c) *There exists a tubular neighborhood T of $W = L - L \cap U$ such that $\tau = \pi^*(\tau | W)$ on T, where $\pi: T \to W$ is the projection.*

Remark. In the above lemma, if $p > r$, then τ' vanishes about L.

Proof. By induction on r, we may assume, for $r \geqslant 0$, that τ vanishes about the closure of a neighborhood U of L_{r-1}. For $\varepsilon > 0$ sufficiently small, let $T(\varepsilon)$ be the ε-tubular neighborhood of $W = L - L \cap U$ in M. Then W is a deformation retract of $T(\varepsilon)$. Let $\pi: T(\varepsilon) \to W$ be the projection.

Let $\dot{W} = L \cap (\text{the boundary of } U) = \text{the boundary of } W$ in $L - L_{r-1}$. Then $\dot{W}$ is compact. We choose ε sufficiently small so that τ vanishes about $\pi^{-1}\dot{W}$. Since π is a homotopy equivalence, the closed p-form $\tau - \pi^*(\tau | W)$ is exact about $T(\varepsilon)$. There exists a $(p-1)$-form θ defined about $T(\varepsilon)$ and vanishing about $\pi^{-1}\dot{W}$ such that $d\theta = \tau - \pi^*(\tau|W)$ on $T(\varepsilon)$. In fact, θ can be obtained by using the Poincaré lemma based on the deformation retraction of the tubular neighborhood $T(\varepsilon)$.

Let ϕ be a C^∞ function on $T(\varepsilon)$ such that $\phi = 0$ on $T(\varepsilon) - T(\frac{1}{2}\varepsilon)$ and $=1$ on $T(\frac{1}{4}\varepsilon)$. Let $\theta' = \phi\theta$ on $T(\varepsilon)$ and $=0$ on $M - T(\varepsilon)$. Then θ' is a well-defined $(p-1)$ form on M. Moreover $\tau' = \tau - d\theta'$ vanishes about U and $\tau' = \pi^*(\tau|W)$ on $T(\frac{1}{4}\varepsilon)$. Hence the lemma is proved.

2. Poincaré Duality and a Residue Formula

We shall explain the meaning of Poincaré duality between a closed form and an oriented stratified C^∞ set L and give a residue formula for the intersection number with L.

Let L be an oriented r-dimensional stratified C^∞ set in an oriented C^∞ manifold M. Let $q = \dim M - r$ and assume that $q \geqslant 1$.

A smooth q-simplex $\beta: \Delta^q \to M$ is said to be transversal to the stratified C^∞ set L if (a) $\beta^{-1}L$ lies in the interior of Δ^q with $\beta^{-1}L_{r-1} = \varnothing$ and (b) β is transversal to $L - L_{r-1}$.

A smooth simplicial q-chain $\mathfrak{z}$ is transversal to L if each simplex involved in $\mathfrak{z}$ is transversal to L.

Definition. Let w be a closed q-form on M. An oriented stratified C^∞ set L of codimension q in M is said to be Poincaré dual to w if, for any smooth simplicial q-cycle $\mathfrak{z}$ transversla to L in M, the intersection number of $\mathfrak{z}$ and L is equal to the integral $\int_{\mathfrak{z}} w$. Moreover, there exists a smooth simplicial q-cycle transversal to L with $\int_{\mathfrak{z}} w \neq 0$.

Thus, it is automatically assumed that the cohomology class $[w]$ of w is nontrivial.

The above definition of Poincaré duality is in conformity with the usual definition when L is either a submanifold or an analytic subvariety.

We also observe that, since $\int_{\mathfrak{z}} w = 0$ for any smooth q-cycle $\mathfrak{z}$ lying in $M - L$, w must be exact on $M - L$.

Let U be a neighborhood of 0 in R^q and let $f: U \to M$ be a C^∞ map with $f^{-1}L = \{0\}$ and $f(0) \in L - L_{r-1}$. Let W be a neighborhood of $f(0)$ in $L - L_{r-1}$ having a tubular neighborhood T in M with a product structure $T = V \times W$, where V is a q-ball with center 0. We may assume that $fU \subset T$. Then the intersection number of f and L is the degree of the homomorphism

$$H_q(U, U - \{0\}) \to H_q(V, V - \{0\})$$

induced by the composite map $\hat{f}$:

$$(U, U-\{0\}) \xrightarrow{f} (T, T-W) \xrightarrow{\text{projection}} (V, V-\{0\}).$$

LEMMA 2.1. *If v is a $(q-1)$-form on $M-L$ such that $dv = w|M-L$, then*

$$-\lim_{\varepsilon\to 0}\int_{\partial B(\varepsilon)} f^*v = \textit{the intersection number of } f \textit{ and } L, \tag{2.1}$$

where $B(\varepsilon)$ denotes the open ε-ball about 0 *in R^q with the orientation induced by that of R^q.* It is assumed that $L-L_{r-1}$ is connected.

Proof. The existence of the r.h.s. of (2.1) follows from the fact that

$$\int_{\partial B(\varepsilon_2)} f^*v - \int_{\partial B(\varepsilon_1)} f^*v = \int_{B(\varepsilon_2)-B(\varepsilon_1)} f^*w$$

tends to 0 as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$. It can be verified that the r.h.s. of (2.1) does not change under a C^∞ homotopy of the map

$$f: (U, U-\{0\}) \to (M, M-L)$$

provided the point $f(0)$ remains in $L-L_{r-1}$ during the homotopy.

We now consider the special case where the map f is transversal to $L-L_{r-1}$ so that the intersection number of f and L is equal to 1. Set

$$c = \lim_{\varepsilon\to 0}\int_{\partial B(\varepsilon)} f^*v.$$

Let $\mathfrak{z} = \sum m_i\beta_i$ be a smooth q-cycle of M transversal to L with $\int_{\mathfrak{z}} w \neq 0$. Replacing each smooth simplex by a subdivision if necessary, we may assume that $\beta_i^{-1}L$ consists at most of one sngle point ξ_i in the interior of Δ^q so that the intersection number v_i of β_i and L is either ± 1 or 0.

Let $B_i(\varepsilon)$ denote the open ε-ball about ξ_i in Δ^q. Then

$$\sum m_i v_i = \text{the intersection number of } \mathfrak{z} \text{ and } L = \int_{\mathfrak{z}} w$$

$$= \sum m_i \lim_{\varepsilon\to 0}\int_{\Delta^q - B_i(\varepsilon)} \beta_i^* w = -\sum m_i \lim_{\varepsilon\to 0}\int_{\partial B_i(\varepsilon)} \beta_i^* v = -\sum m_i v_i c.$$

We conclude that $c = -1$.

Let V be the q-ball as before. The inclusion map g:

$$(V, V-\{0\}) = (V, V-\{0\}) \times f(0) \subset (T, T-W) \subset (M, M-L)$$

is transversal to L so that $\lim_{\varepsilon\to 0}\int_{\partial B(\varepsilon)} g^*v = -1$. Now the map

$$f\colon (U, U-\{0\}) \to (M, M-L)$$

is homotopic to the composite map

$$(U, U-\{0\}) \xrightarrow{\hat{f}} (V, V-\{0\}) \xrightarrow{g} (M, M-L).$$

Hence

$$\lim_{\varepsilon\to 0}\int_{\partial B(\varepsilon)} f^*v = \eta\left(\lim_{\varepsilon\to 0}\int_{\partial B(\varepsilon)} g^*v\right) = -\eta.$$

where η is the intersection number of f and L.

3. Poincaré Duality under a C^∞ Map

Let M and N be oriented C^∞ manifolds, and let $f\colon N\to M$ be a C^∞ map. Let L be an oriented stratified C^∞ set of codimension q in M such that $\Gamma = f^{-1}L$ has a C^∞ stratification

$$\Gamma = \Gamma_p \supset \Gamma_{p-1} \supset \cdots \supset \Gamma_0$$

of codimension q in N. We assume that $f^{-1}L_{r-1} \subset \Gamma_{p-1}$ where $r = \dim M - q$.

By a transversal slice B of Γ at a point y of $\Gamma - \Gamma_{p-1}$, we mean a q-dimensional subminifold of N, which meets Γ transversally at the point y only.

Definition. Let B be a transversal slice of Γ at $y \in \Gamma - \Gamma_{p-1}$. We define $\operatorname{ord}_f(\Gamma, y)$ to be the intersection number of $f|B$ and L. The orientation of B is so chosen that $\operatorname{ord}_f(\Gamma, y) \geqslant 0$.

This integer $\operatorname{ord}_f(\Gamma, y)$ is independent of the choice of the transversal slice B and is locally constant on $\Gamma - \Gamma_{p-1}$.

Let $\Gamma - \Gamma_{p-1}$ consists of a finite number of connected components $\Gamma(1),\dots,\Gamma(l)$. Define

$$\operatorname{ord}_f \Gamma(i) = \operatorname{ord}_f(\Gamma, y)$$

fr an arbitrary point y of $\Gamma(i)$. Let B be a transversal slice of Γ at y, and let B be oriented as in the definition of $\operatorname{ord}_f(\Gamma, y)$. If $\operatorname{ord}_f \Gamma(i) > 0$, then the orientation of B is well-defined, and $\Gamma(i)$ has an orientation determined by those of N and B so that the orientation of N about y is the product orientation of B and $\Gamma(i)$.

THEOREM 3.1. *Let $f: N \to M$ be a C^∞ map and let L and $\Gamma = f^{-1}L$ be stratified C^∞ sets as described above. Let L be Poincaré dual to a closed q-form w on M. Let the connected components $\Gamma(1), \ldots, \Gamma(l)$ of $\Gamma - \Gamma_{p-1}$ be such that the integral* $(\mathrm{ord}_f \Gamma(i)) \int_{\Gamma(i)} \sigma$ *is summable for any p-form σ on N. If N is compact, then, for any closed p-form σ on N,*

$$\int_N f^* w \wedge \sigma = \sum_{1 \leqslant i \leqslant l} (\mathrm{ord}_f \Gamma(i)) \int_{\Gamma(i)} \sigma. \tag{3.1}$$

Remark 1. The theorem is known in the case where f is transversal to a submanifold L of M. In this case, each $\Gamma(i)$ is a submanifold of codimension q in N with $\mathrm{ord}_f \Gamma(i) = 1$. The current version of this theorem arises from a more restrictive version announced in [2].

Remark 2. If each $\Gamma(i)$ is a closed submanifold, the formula (3.1) means that the cycle $\sum (\mathrm{ord}_f \Gamma(i)) \Gamma(i)$ is Poincaré dual to $f^* w$.

Proof. According to Lemma 1.1, we may assume that σ vanishes about the closure of a neighborhood U of Γ_{p-1} in N. Let $W = \Gamma - \Gamma \cap U$. We may further assume that σ coincides with $\pi^*(\sigma | W)$ on some ε-tubular neighborhood $T(\varepsilon)$ of W, where π: $T(\varepsilon) \to W$ is the projection.

Let $W_i = \Gamma(i) - \Gamma(i) \cap U$. Let $T_i(\varepsilon)$ be the ε-tubular neighborhood of W_i in N. By a choice of sufficiently small ε, all $T_i(\varepsilon)$ are well defined and mutually disjoint. Of course, $T(\varepsilon) = \bigcup T_i(\varepsilon)$.

Let v be a closed $(q-1)$-form on $M - L$ so that $dv = w | M - L$. Then, on $N - \Gamma$,

$$d(f^* v \wedge \sigma) = f^* w \wedge \sigma.$$

We choose a sufficiently fine smooth triangulation of N so that σ vanishes on each simplex that meets U. Let K' (resp. K'') be the union of all simplices that meet $T(\varepsilon)$ (resp. U). Since σ vanishes on K'', we have

$$\int_{N - K' \cup K''} f^* w \wedge \sigma = -\int_{\partial K'} f^* v \wedge \sigma.$$

Through making the triangulation arbitrarily fine, we conclude that

$$\int_{N - T(\varepsilon) \cup U} f^* w \wedge \sigma = -\sum_i \int_{\dot{T}_i(\varepsilon)} f^* v \wedge \sigma$$

where $\dot{T}_i(\varepsilon)$ denotes the "ε-tube" 9f W_i. It follows that

$$\int_N f^* w \wedge \sigma = \lim_{\varepsilon \to 0} \int_{N - T(\varepsilon) \cup U} f^* w \wedge \sigma = -\lim_{\varepsilon \to 0} \sum \int_{\dot{T}_i(\varepsilon)} f^* v \wedge \sigma$$

$$= -\sum \lim_{\varepsilon \to 0} \int_{\dot{T}_i(\varepsilon)} f^* v \wedge \pi^*(\sigma | W).$$

The integration is first carried out along each fiber of $\dot{T}_i(\varepsilon)$. The fiber $B(\varepsilon, y)$ of $T_i(\varepsilon)$ above $y \in W_i$ is a transversal slice to Γ. According to Lemma 2.1, $\lim_{\varepsilon\to 0} \int_{\partial B(\varepsilon, y)} f^*v = -\mathrm{ord}_f \Gamma(i)$. We have

$$\lim_{\varepsilon\to 0} \int_{\dot{T}_i(\varepsilon)} f^*v \wedge \pi^*(\sigma | W) = -(\mathrm{ord}_f \Gamma(i)) \int_{\Gamma(i)} \sigma.$$

Hence the theorem is proved.

Since an analytic subvariety has a natural stratification through successive singular loci, many of the conditions in the above theorem are automatically satisfied in the complex analytic case. A holomorphic version of Theorem 3.1 can be stated as follows.

THEOREM 3.1A. *Let M and N be complex analytic manifolds, of which N is compact. Let L be an analytic subvariety Poincaré dual to a closed $2k$-form w on M. Let $f: N \to M$ be a holomorphic map such that the analytic subvariety $\Gamma = f^{-1}L$ is of codimension $\geqslant k$ in N. Let $\Gamma(1), \ldots, \Gamma(l)$ be the irreducible components of Γ, which are of codimension k in N. If L_s is the singular locus of L and if $f^{-1}L_s$ is of codimension $>k$ in N, then, for any closed $2p$-form σ on N, $p + q = \dim N$,*

$$\int_N f^*w \wedge \sigma = \sum (\mathrm{ord}_f \Gamma(i)) \int_{\Gamma(i)} \sigma.$$

where the orientation of each irreducible component $\Gamma(i)$ is induced by its complex structure.

Proof. The theorem is immediate if Δ is of codimension $>k$ in N. Assume that Δ is of codimension k, i.e., of real dimension $2p$. Let Δ_{2p-2} be the union of Δ' and the singular locus of Δ. Then Δ_{2p-2} is an analytic subvariety of N and has a natural stratification $\Delta_{2p-2} \supset \Delta_{2p-4} \supset \cdots$ consisting of successive singular loci. Thus, $\Delta \supset \Delta_{2p-2} \supset \Delta_{2p-4} \supset \cdots$ is a stratification of Δ. Moreover, the connectedness of the smooth part of $\Delta(i)$ is not changed by removing points belonging to Δ'.

By taking a holomorphic transversal slice in order to determine $\mathrm{ord}_f \Gamma(i)$, we conclude that the orientation of $\Gamma(i)$ for the integral at the r.h.s. of (3.1) must be induced by the complex structure of $\Gamma(i)$ so that the orientation of the complex manifold N is respected. Hence the theorem is proved.

4. DEGENERACY INDICES

Let N be an oriented C^∞ manifold, and let $\pi: E \to N$ be a C^∞ complex k-plane bundle.

First, we consider the case where N is of dimension $2k$, which is the real dimension of the fiber of E. Let y_0 be an isolated zero of a C^∞ section s of E, and let $B(\varepsilon)$ denote an ε-ball about y_0. Then the index of s at y_0 is the degree of the map $S^{2k-1} = \partial B(\varepsilon) \to C^k - \{0\} \to S^{2k-1}$ induced by the composite map

$$B(\varepsilon) \xrightarrow{s} \pi^{-1}(B(\varepsilon)) \xrightarrow{\text{trivialization}} C^k \times B(\varepsilon) \xrightarrow{\text{projection}} C^k. \tag{4.1}$$

In general, let $s_1, ..., s_r$ be C^∞ sections of E. Let Δ be the degeneracy set of $s_1, ..., s_r$, i.e., the set

$$\Delta = \{y \in N; s_1 y, ..., s_r y \text{ linearly dependent}\}.$$

Set $k' = k - r + 1$. Let y_0 be a point of Δ such that (a) $s_1 y_0, ..., s_r y_0$ are of rank $r - 1$ and (b) about y_0, Δ is locally a C^∞ submanifold of codimension $2k'$ in N. Let us assume that $s_1 y_0, ..., s_{r-1} y_0$ are linearly independent. Let B be a transversal slice of Δ at y_0 and let $\hat{E}$ denote the quotient complex k'-plane bundle $E/(s_1, ..., s_{r-1})$ restricted to B. Then $s_r|B$ represents a C^∞ section of $\hat{E}$. We define

$$\text{Index}_{s_1,\ldots,s_r}(\Delta, y_0) = \text{the index of } s_r|B \text{ of } \hat{E} \text{ at } y_0. \tag{4.2}$$

An orientation of B is chosen so that

$$\text{Index}_{s_1,\ldots,s_r}(\Delta, y_0) \geqslant 0.$$

This definition does not depend on the choice of the transversal slice B, and the index is locally constant about y_0 in Δ.

We claim that the definition of $\text{Index}_{s_1,\ldots,s_r}(\Delta, y_0)$ does not depend on the particular choice of s_r. It suffices to verify that, if $s_2 y_0, ..., s_r y_0$ are linearly independent, then

$$\text{Index}_{s_2,\ldots,s_r,s_1}(\Delta, y_0) = \text{Index}_{s_1,\ldots,s_r}(\Delta, y_0).$$

For this purpose, we write $s_1 y_0$ as a linear combination $c_2 s_2 y_0 + \cdots + c_r s_r y_0$ where $c_r \neq 0$. Set $s_1(\theta) = s_1 \cos\theta + c_r s_r \sin\theta$ and $s_r(\theta) = -s_1 \sin\theta + c_r s_r \cos\theta$. Then the degeneracy set of $s_1(\theta), s_2, ..., s_{r-1}, s_r(\theta)$ is precisely Δ. Since, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, $s_1(\theta) y_0, s_2 y_0, ..., s_{r-1} y_0$ are linearly independent, the integer $\text{Index}_{s_1(\theta),s_2,\ldots,s_{r-1},s_r(\theta)}(\Delta, y_0)$ is independent of θ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. This confirms the claim.

In essence, the next theorem is an improvement of the Gauss–Bonnet Formula II as given in Griffiths and Harris [6] and is also a generalization of the Hopf Index Theorem to Chern classes.

THEOREM 4.1. *Let N be an oriented compact C^∞ manifold, and let E be a complex k-plane bundle over N having C^∞ sections $s_1,\ldots, s_r$, $r \geqslant 1$. Let the degeneracy set of $s_1,\ldots, s_r$ be a stratified C^∞ set Δ of codimension $2k'$ in N with a C^∞ stratification*

$$\Delta = \Delta_p \supset \Delta_{p-1} \supset \cdots \supset \Delta_0, \qquad p + 2k' = \dim N.$$

Let $\Delta(1),\ldots, \Delta(l)$ be the connected components of $\Delta - \Delta_{p-1}$ such that each integral $(\mathrm{Index}_{s_1,\ldots,s_r} \Delta(i)) \int_{\Delta(i)} \sigma$ *is summable for any p-form σ on N. Let $c_{k'}$ denote a closed $2k'$-form on N representing the k'th Chern class of E. If $s_1,\ldots, s_{r-1}$ are linearly independent everywhere in $\Delta - \Delta_{p-1}$ and if σ is any closed p-form on N, then*

$$\int_N c_{k'} \wedge \sigma = \sum_i (\mathrm{Index}_{s_1,\ldots,s_r} \Delta(i)) \int_{\Delta(i)} \sigma, \tag{4.3}$$

where $\mathrm{Index}_{s_1,\ldots,s_r} \Delta(i) = \mathrm{Index}_{s_1,\ldots,s_r}(\Delta, y)$ *for an arbitrary point y of $\Delta(i)$.*

Remark. The l.h.s. of the formula (4.3) is independent of the choices of the sections $s_1,\ldots, s_r$.

Proof. Let E be embedded as a C^∞ subbundle of the product complex vector bundle $C^{n-r} \times N$, for some $n > r$. For each $y \in N$, the fiber E_y can be taken as a k-plane in C^{n-r}. Let $\{e_1,\ldots, e_n\}$ be an orthonormal basis of C^n so that $\{e_1,\ldots, e_{n-r}\}$ becomes a basis for C^{n-r}. The inner product of C^n will be denoted by $\langle\, ,\, \rangle$.

Let M denote the Grassmannian $G(k, C^n)$ of complex k-planes in C^n. Let $L(r)$ be the Schubert variety consisting of all k-planes V with

$$\dim(V \cap C^{n-r}) \geqslant k' = k - r + 1.$$

This implies that $\dim(V \cap C^{n-r-i}) \geqslant k' - i$ for $i = 1,\ldots, k' - 1$ and $\dim(V \cap C^{n-i}) \geqslant k - i$ for $i = 0,\ldots, r - 1$.

According to the convention in [5], the symbol for the Schubert variety $L(r)$ is $(1,\ldots, 1, 0,\ldots, 0)$ with "1" occurring k' times. Therefore, $L(r)$ is Poincaré dual to $(-1)^k$ times the k'th Chern class of the canonical k-plane bundle of M. Let

$$\lambda\colon E \to C^n \times N \tag{4.4}$$

be the vector bundle map such that, for $y \in N$ and $v \in E_y$, λv is the vector

$$v + \lambda_1(v)\, e_n + \lambda_2(v)\, e_{n-1} + \cdots + \lambda_r(v)\, e_{n-r+1},$$

where

$$\lambda_1(v) = \langle s_1 y, v \rangle, \qquad \lambda_2(v) = \begin{vmatrix} \langle s_1 y, s_1 y \rangle & \langle s_2 y, s_1 y \rangle \\ \langle s_1 y, v \rangle & \langle s_2 y, v \rangle \end{vmatrix},$$

$$\vdots$$

$$\lambda_r(\lambda) = \begin{vmatrix} \langle s_1 y, s_1 y \rangle & \cdots & \langle s_r y, s_1 y \rangle \\ \vdots & & \\ \langle s_1 y, s_{r-1} y \rangle & \cdots & \langle s_r y, s_{r-1} y \rangle \\ \langle s_1 y, v \rangle & \cdots & \langle s_r y, v \rangle \end{vmatrix}.$$

Evidently λ is an injective C^∞ map, and $\tilde{E} = \lambda E$ is a k-plane subbundle of $C^n \times N$.

For $y \in N$, let

$$E_y^\# = \{v \in E_y; \langle s_1 y, v \rangle = \cdots = \langle s_r y, v \rangle = 0\}.$$

Then $\dim E_y^\# \geqslant k'$. Since $\lambda_1(E_y^\#) = \cdots = \lambda_r(E_y^\#) = 0$, we have $\lambda E_y^\# = E_y^\# \subset C^{n-r}$.

Each fiber $\tilde{E}_y$ of $\tilde{E}$ is a k-plane in C^n. Let

$$f: N \to M \tag{4.5}$$

be the C^∞ map given by $f(y) = \tilde{E}_y \in M$. We claim that $f(y) \in L(r)$ if and only if $y \in \Delta$. The sufficiency is easy to see. In fact, if $y \in \Delta$, then $s_1 y, \ldots, s_r y$ are of rank $< r$ so that $\dim E_y^\# \geqslant k - r + 1$. Thus,

$$\dim(\tilde{E}_y \cap C^{n-r}) \geqslant \dim E_y^\# \geqslant k'.$$

In order to see the necessity part of the claim, let $f(y) \in L(r)$, which means that $\dim(\tilde{E}_y \cap C^{n-r}) \geqslant k'$. If $s_1 y, \ldots, s_r y$ were linearly independent, then the $r \times r$ hermitian matrix $[\langle s_i y, s_j y \rangle]$ must be nonsingular, and all of its principal minors must be nonzero. If $v \in \tilde{E}_y \cap C^{n-r}$, then $\lambda v = v$ and $\lambda_1(v) = \cdots = \lambda_r(v) = 0$. It follows that $\langle s_1 y, v \rangle = \cdots = \langle s_r y, v \rangle = 0$, and

$$\tilde{E}_y \cap C^{n-r} = \{v \in E_y; \langle s_1 y, v \rangle = \cdots = \langle s_r y, v \rangle = 0\} = E_y^\#.$$

This leads to the contradiction that

$$\dim(\tilde{E}_y \cap C^{n-r}) = k - r < k'.$$

Therefore, $s_1 y, \ldots, s_r y$ must be linearly dependent, and $y \in \Delta$.

It remains to show that, for $y_0 \in \Delta - \Delta_{p-1}, f(y_0)$ is a smooth point of $L(r)$, and

$$\mathrm{Index}_{s_1, \ldots, s_r}(\Delta, y_0) = \overline{\mathrm{ord}}_f(\Delta, y_0). \tag{4.6}$$

Then the proof can be completed by applying Theorem 3.1.

The Schubert variety $L(r)$ has a stratification through successive singular loci. The associated Schubert cell $\mathfrak{S}$ consists of all k-planes V in C^n such that $\dim(V \cap C^{n-r-i}) = k' - i$, $1 \leqslant i \leqslant k' - 1$, and $\dim(V \cap C^{n-i}) = k - i$, $0 \leqslant i \leqslant r - 1$. Every point of $\mathfrak{S}$ is a smooth point of $L(r)$. If the first $n - r$ basis elements $\{e_1, \dots, e_{n-r}\}$ of C^n undergo a permutation, the Schubert variety $L(r)$ remains unchanged, but the Schubert cell $\mathfrak{S}$ is no more the same.

Since $s_1 y_0, \dots, s_{r-1} y_0$ are linearly independent, we have $\lambda_i(s_i y_0) \neq 0$ and $\lambda_{i+1}(s_i y_0) = \cdots = \lambda_r(s_i y_0) = 0$, $1 \leqslant i \leqslant r - 1$. The k-plane $\tilde{E}_{y_0}$ is generated by k' vectors in C^{n-r} and $\lambda(s_1 y_0), \dots, \lambda(s_{r-1} y_0)$. The latter are linearly independent modulo C^{n-r+1}. We may represent $\tilde{E}_{y_0}$ by a $k \times n$ echelon matrix of the type $\left[\begin{smallmatrix} * & 0' \\ * & 0'' I_{r-1} \end{smallmatrix}\right]$ where $0'$ (resp. $0''$) denotes the $k' \times r$ (resp. $(r-1) \times 1$) zero matrix. Evidently $f(y_0) = \tilde{E}_{y_0}$ belongs to some Schubert cell obtained from $\mathfrak{S}$ through some permutation of $\{e_1, \dots, e_{n-r}\}$ and is therefore a smooth point of $L(r)$.

5. Proof of the Formula (4.6)

First we are going to compute the index at the l.h.s. of (4.6). For y about y_0, there is an orthogonal direct sum decomposition

$$E_y = E'_y \oplus (s_1 y, \dots, s_{r-1} y),$$

where $(s_1 y, \dots, s_{r-1} y)$ denotes the $(r-1)$-dimensional subspace of E_y spanned by $s_1 y, \dots, s_{r-1} y$. This gives rise to a direct sum decomposition

$$s_r = s'_r + s''_r$$

of the section s_r locally about y_0. Denote by E' the k'-plane bundle over a neighborhood of y_0 with E'_y as the fiber above y. Then s'_r is a section of E' and corresponds to the section s_r of the quotient bundle $E/(s_1, \dots, s_{r-1})$. Let B be a transversal slice of Δ at y_0. Then $\mathrm{Index}_{s_1,\dots,s_r}(\Delta, y_0)$ is simply the index of the section s' of $E' | B$ at y_0.

Let $v_1, \dots, v_{k'}$ be orthonormal local sections of E' about y_0 so that $v_1(y), \dots, v_{k'}(y)$ from an orthonormal basis of E'_y for y about y_0. Then $\mathrm{Index}_{s_1,\dots,s_r}(\Delta, y_0)$ is the degree of the map

$$S^{2k'-1} = \partial B(\varepsilon) \to S^{2k'-1} \tag{5.1}$$

given by $y \mapsto z''(y)/\|z''(y)\|$, where $z''(y) = (z''_1(y), \dots, z''_{k'}(y))$ and

$$z''_i(y) = \langle v_i(y), s'_r y \rangle.$$

A direct computation shows that $\lambda(v_i(y)) = v_i(y) + \lambda_r(v_i(y))\, e_{n-r+1}$ and

$$\lambda_r(v_i(y)) = \lambda_{r-1}(s_{r-1}\, y)\, \bar{z}_i''(y). \tag{5.2}$$

Since $s_r y_0$ i a linear combination of $s_1 y_0, \dots, s_{r-1} y_0$, we have $\lambda_r(v_i(y_0)) = 0$, i.e.,

$$\lambda(v_i(y_0)) = v_i(y_0).$$

In order to compute $\operatorname{ord}_f(\Delta, y_0)$, we choose the orthonormal basis $\{e_1, \dots, e_n\}$ of C^n such that $\{e_{n-k}, \dots, e_{n-r}\}$ is a basis of E'_{y_0}. Then, about $f(y_0)$, M has local coordinates $\zeta = (\zeta', \zeta'')$ with $\zeta' = (\zeta'_{ij})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n-k-1}$ and $\zeta'' = (\zeta_i'')_{1 \leqslant i \leqslant k}$ such that ζ represents the k-plane spanned by

$$e_{n-k+i-1} + \sum_j \zeta'_{ij} e_j + \zeta_i'' e_{n-r+1}, \qquad 1 \leqslant i \leqslant k'.$$

$$e_{n-k-i} + \sum_j \zeta'_{ij} e_j + \zeta_i'' e_{n-r+1}, \qquad k' < i \leqslant k.$$

Thus, $L(r)$ is locally given by the equations $\zeta_1'' = \cdots = \zeta_{k'}'' = 0$. Moreover $\zeta''(f(y_0)) = 0$. We may take $\zeta_1'', \dots, \zeta'_{k'}$ as local coordinates of the transversal slice D of $L(r)$ given by the equations $\zeta' = \zeta'(f(y_0))$ and $\zeta''_{k'+1} = \cdots = \zeta_k'' = 0$.

Let B be the transversal slice of Δ at y_0 as before. Then $\operatorname{ord}_f(\Delta, y_0)$ is the intersection number of $f|B$ and $L(r)$, which is equal to the degree of the composite map

$$S^{2k'-1} = \partial B(\varepsilon) \to \text{a deleted neighborhood of} f(y_0) \text{ in } M - L$$

$$\xrightarrow{\text{projection}} D - \{f(y_0)\} \to S^{2k'-1}, \tag{5.3}$$

where the projection is givey by $(\zeta', \zeta'') \mapsto (\zeta_1'', \dots, \zeta_{k'}'') = \zeta^{\#}$. Therefore, the map (5.3) is given by

$$y \mapsto \zeta^{\#}(f(y))/\|\zeta^{\#}(f(y))\|.$$

Now $f(y) = \tilde{E}_y$ has as a basis:

$$\lambda(v_1(y)), \dots, \lambda(v_{k'}(y)), \lambda(s_1\, y), \dots, \lambda(s_{r-1}\, y),$$

which are written as row vectors of a $k \times n$ matrix. Thus,

$$\lambda(v_i(y)) = (\dots, \lambda_r(v_i(y)), 0, \dots, 0)$$

and

$$\lambda(s_i\, y) = (\dots, 0, \dots, 0, \lambda_i(s_i\, y), \dots, \lambda_1(s_i\, y)),$$

and the matrix has the appearance of

$$\begin{bmatrix} * & A & Z & 0 \\ * & * & 0 & C \end{bmatrix},$$

where (a) $A = A(y)$ is a $k' \times k'$ matrix, (b) $Z = Z(y)$ is the $k' \times 1$ matrix with $Z_i = \lambda_r(v_i(y))$, and (c) $C = C(y)$ is the $(r-1) \times (r-1)$ matrix

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \lambda_1(s_1 y) \\ 0 & \cdots & 0 & \lambda_2(s_2 y) & \lambda_1(s_2 y) \\ & \vdots & & & \\ \lambda_{r-1}(s_{r-1} y) & & \cdots & & \lambda_1(s_{r-1} y) \end{bmatrix}.$$

We may demand that $v_i(y_0) = e_{n-k+i-1}$, $1 \leqslant i \leqslant k'$, so that $A(y_0)$ is the identity $k' \times k'$ matrix. Since, for y near y_0, both A and C are nonsingular, we conclude that the k-plane $f(y)$ has a basis consisting of the row vectors of a $k \times n$ matrix of the type

$$\begin{bmatrix} * & I_{k'} & A^{-1}Z & 0 \\ * & * & 0 & I_{r-1} \end{bmatrix}.$$

This means that $\zeta^{\#}(f(y)) = A(y)^{-1} Z(y)$. For ε sufficiently small, the map (5.3) is homotopic to the map given by $y \mapsto Z(y)/\|Z(y)\|$, which, according to (5.2) is equal to $\bar{z}''(y)/\|z''(y)\|$. The degree of this map differs from that of the map (5.1) only by a factor of $(-1)^{k'}$, and the formula (4.6) is established.

Since both $\mathrm{Index}_{s_1,\ldots,s_r}(\Delta, y_0)$ and $\mathrm{ord}_f(\Delta, y_0)$ are nonnegative, we conclude that the respective orientations of the transversal slice B for determining these two integers must differ by $(-1)^{k'}$, and so do those of $\Delta(i)$. Let each $\Delta(i)$ be oriented as in the determination of $\mathrm{Index}_{s_1,\ldots,s_r}\Delta(i)$. Applying Theorem 3.1, we have

$$\int_N (-1)^{k'} c_{k'} \wedge \sigma = \sum (\mathrm{ord}_f \Delta(i)) \int_{(-1)^{k'}\Delta(i)} \sigma$$

Hence the theorem is proved.

6. Analytic Versions of the Main Theorem

Let $\pi\colon E \to N$ be a holomorphic k-plane bundle. If the dimension of the complex manifold N is also equal to k, the index of a holomorphic section of E at an isolated zero y_0 is always positive. In fact, the composite map (4.1) is a holomorphic map, which preserves orientation. The image of $B(\varepsilon)$ in C^k has a positive algebraic volume.

In general, let $s_1,\ldots, s_r$ be holomorphic sections of E with $k' = k-r+1 \geqslant 0$. Let Δ be the degeneracy set of $s_1,\ldots, s_r$, which is an analytic subvariety of N. Let y_0 be a point of Δ such that (a) $s_1 y_0,\ldots, s_{r-1} y_0$ are linearly independent, and (b) about y_0, Δ is locally a holomorphic submanifold of codimension k' in N. Let B be a holomorphic transversal slice of Δ at y_0. Then $\hat{E} = E/(s_1,\ldots, s_{r-1})$ is a holomorphic k'-plane bundle over B, provided B is sufficiently small. As a holomorphic section of $\hat{E}$, $s_r|B$ has an isolated zero y_0 in B. Thus,

$$\mathrm{Index}_{s_1,\ldots,s_r}(\Delta, y_0)$$

as defined in (4.2), is always positive, while the orientation of B in induced by the complex structure of B.

A holomorphic version of Theorem 4.1 is as follows.

THEOREM 6.1. *Let N be a compact complex manifold, and let E be a holomorphic k-plane bundle over N having holomorphic sections $s_1,\ldots, s_r$. Assume that the degeneracy set Δ (resp. Δ') of $s_1,\ldots, s_r$ (resp. $s_1,\ldots, s_{r-1}$) is of codimension $\geqslant k'$ (resp. $>k'$). If $c_{k'}$ is a closed form on N representing the* k'th *Chern class of E, then, for every $2p$-form σ on N with $p+k' = \dim N$,*

$$\int_N c_{k'} \wedge \sigma = \sum_i (\mathrm{Index}\, \Delta(i)) \int_{\Delta(i)} \sigma \tag{6.1}$$

where $\mathrm{Index}\, \Delta(i) = \mathrm{Index}_{s_1,\ldots,s_r}(\Delta, y)$ *for some smooth point y of $\Delta(i)$ not in Δ', and the orientation of each $\Delta(i)$ is induced by the complex structure.*

The proof of this theorem is the same as that of Theorem 3.1A.

As we have mentioned in the Introduction, the above theorem is a known result of Chern. Our only contribution in the formula (6.1) is the explicit definition of the multiplicity, namely, the index.

There is a real analytic version of Theorem 4.1 as follows.

THEOREM 6.2. *Let E be a complex k-plane bundle real analytic over a compact oriented real analytic n-manifold N. Let Δ* (resp. *Δ') be the degeneracy set of real analytic sections $s_1,\ldots, s_r$ (resp. $s_1,\ldots, s_{r-1}$) of E. Let Δ (resp. Δ') be of codimension $\geqslant 2k'$ (resp. $>2k'$) in N. Let Δ_s be the singular locus of Δ, which consists of those points, about which Δ is locally not a $(n-2k')$-dimensional real analytic submanifold of N. Let the connected components $\Delta(1),\ldots, \Delta(l)$ of $\Delta - \Delta' \cup \Delta_s$ be such that each* $(\mathrm{Index}\, \Delta(i)) \int_{\Delta(i)} \sigma$ *is summable for any p-form σ on N. If $c_{k'}$ is a closed $2k'$-form on N*

representing the k'th *Chern class of* E, *and if* σ *is any closed* $(n-2k')$*-form on* N, *then*

$$\int_N c_{k'} \wedge \sigma = \sum_i \operatorname{Index} \Delta(i) \int_{\Delta(i)} \sigma.$$

Proof. Since every real analytic variety has a stratification by successive singular loci, this theorem is essentially a corollary of Theorem 4.1. One needs just to verify that $\Delta - \Delta' \cup \Delta_s$ has only a finite number of connected components. This follows from the fact that Δ is the continuous image of a finite simplicial complex K such that (a) the interior of each simplex of K is mapped homeomorphically into Δ and (b) a subcomplex K' of K is mapped onto $\Delta' \cup \Delta_s$. As matter of fact, a local version of this result given in Koopman and Brown [8] is already adequate for our purpose.

Remark. It seems that the summability condition on the integral $(\operatorname{Index} \Delta(i)) \int_{\Delta(i)}$ is nonessential.

7. Pontryagin Classes

Let E be a C^∞ real k-plane bundle over a compact oriented manifold N. Then the ith Pontryagin class $p_i(E)$ is equal to $(-1)^i$ times the $2i$th Chern class $c_{2i}(E \otimes C)$ of the complexification $E \otimes C$ of E. If s'_μ and s''_μ, $\mu = 1,\ldots, r$, are C^∞ sections of E, then $s_\mu = s'_\mu + is''_\mu$, $\mu = 1,\ldots, r$, are C^∞ sections of $E \otimes C$. In this sense, we remark that Theorem 4.1 is applicable to Pontryagin classes.

We illistrate this remark with the special case where $r = 1$. Let s' and s'' be C^∞ sections of E with respective zero sets Δ' and Δ''. Then $\Delta = \Delta' \cap \Delta''$ is the zero set of the section $s = s' + is''$ of $E \otimes C$. For simplicity, let us assume that Δ is the disjoint union of closed submanifolds of N of various codimensions $\geqslant 2k$ in N. Let $\Delta(1),\ldots, \Delta(l)$ be those components of Δ of codimension precisely equal to $2k$. Then, according to Theorem 4.1, the cycle $\sum (\operatorname{Index} \Delta(i)) \Delta(i)$ modulo torsion bounds when k is odd and is Poincaré dual to $(-1)^{k/2} p_{k/2}(E)$ when k is even.

8. A Generalized Bezout Theorem

Let $[z] = [z_0,\ldots, z_n]$ denote the homogeneous coordinates of the complex projective n-space CP^n. Let $\{U_i\}$ be the open covering of CP^n with $U_i = \{[z] \in CP^n; z_i \neq 0\}$. We considers polynomials with complex coefficients.

A polynomial $g(z, \bar{z})$ homogeneous of degree d' in $z = (z_0, \dots, z_n)$ and of degree d'' in $\bar{z} = (\bar{z}_0, \dots, \bar{z}_n)$ gives rose to a polynomial function

$$g(z/z_i, \bar{z}/\bar{z}_i) = g(z, \bar{z})/z_i^{d'}\bar{z}_i^{d''}$$

on each open set U_i. (We shall say that $g(z, \bar{z})$ is a homogeneous polynomial of bidegree $(d'd'')$ in (z, z').) On $U_i \cap U_j$, we have

$$g(z/z_j, \bar{z}/\bar{z}_j) = (z_i/z_j)^{d'} (\bar{z}_i/\bar{z}_j)^{d''} g(z/z_i, \bar{z}/\bar{z}_i).$$

This means that every homogeneous polynomial of bidegree (d', d'') in $(z, \bar{z})$ can be regarded as a section of the line bundle $H^{d'}\bar{H}^{d''}$, whose transition function on $U_1 \cap U_j$ is $(z_i/z_j)^{d'} (\bar{z}_i/\bar{z}_j)^{d''}$. As matter of fact, $H^{d'}\bar{H}^{d''}$ is the tensor product of powers of the hyperplane bundle H and its conjugate $\bar{H}$. If $g_1(z, \bar{z}), \dots, g_k(z, \bar{z})$ are homogeneous polynomials of respective bidegree $(d_1', d_i''), \dots, (d_k', d_k'')$, then the vector $(g_1(z, \bar{z}), \dots, g_k(z, \bar{z}))$ represents a section of the complex k-plane bundle

$$E = H^{d_1'}\bar{H}^{d_1''} \oplus \cdots \oplus H^{d_k'}\bar{H}^{d_k''}. \tag{8.1}$$

The classical Bezout theorem states that if the common zero set of n homogeneous polynomials in z only is finite, then, with multiplicities counted, the number of common zeroes is precisely equal to the product of degrees of these n polynomials. The next assertion is a generalization to a matrix of polynomials homogeneous in both z and $\bar{z}$.

THEOREM 8.1. *Let $f(z, \bar{z}) = (f_{ij}(z, \bar{z}))$ be an $r \times k$ matrix, $r \leqslant k$, such that each entry $f_{ij}(z, \bar{z})$ is a homogeneous polynomial of bidegree (d_j', d_j'') in $(z, \bar{z})$. Let Δ (resp. Δ') be the real algebraic subvariety of CP^n consisting of those points, at which the r row vectors (resp. the first $r-1$ row vectors) of the matrix $f(z, \bar{z})$ are linearly dependent. Set $k' = k - r + 1$. Assume that Δ is of codimension $\geqslant 2k'$ and Δ', of codimension $> 2k'$. Let Δ_s denote the singular locus of Δ, which consists of those points of Δ, about which Δ is locally not a $2k'$-codimensional real analytic submanifold of CP^n. If the connected components $\Delta(1), \dots, \Delta(l)$ of $\Delta - \Delta' \cup \Delta_s$ are such that each* $(\operatorname{Index} \Delta(i)) \int_{\Delta(i)}$ *is summable for any $2(n-k')$-form σ on N, then, for any closed 2-form ω on CP^n representing the generator of $H^2(CP^n; Z)$,*

$$\sum_i \operatorname{Index} \Delta(i) \int_{\Delta(i)} \omega^{n-k'} = C_{k'}, \tag{8.2}$$

where

(a) *the constant $C_{k'}$ is the coefficient of the $t^{k'}$ term of the polynomial*

$$(1 + (d_1' - d_1'')\, t) \cdots (1 + (d_k' - d_k'')\, t);$$

(b) *with the row vectors* $s_1,\dots, s_r$ *of* $f(z,\bar{z})$ *taken as sections of the complex k-plane bundle E in* (8.1), $\operatorname{Index}\Delta(i) = \operatorname{Index}_{s_1,\dots,s_r}\Delta(i)$.

Remark 1. If $k' = n$, i.e., $k - r = n - 1$, then both Δ' and Δ_s are empty. In this case, the theorem asserts that, with multiplicities counted, the degeneracy set Δ, if finite, consists of $C_{k'}$ points. Furthermore, when $r = 1$, the theorem says that the common zero set of n homogeneous polynomial of respective bidegree $(d_1', d_1''),\dots, (d_n', d_n'')$, if finite, consists of $(d_1' - d_1'') \cdots (d_n' - d_n'')$ points in CP^n.

Remark 2. The summability condition on each $\Delta(i)$ holds in many cases, e.g., when Δ and Δ' are complex analytic subvarieties or when Δ and Δ' are both smooth.

Proof. Let ω be the Kähler (1, 1)-form of CP^n, which is normalized in order to represent a generator of the integral cohomology group $H^2(CP^n)$. First we compute the first Chern class of the line bundle $H^{d'}\bar{H}^{d''}$ over CP^n. The zero set of the holomorphic section $s = z_1^{d'}\bar{z}_1^{d''}$ is the hyperplane given by $z_1 = 0$. The projective line $z_2 = \cdots = z_n = 0$ is a transversal slice to this hyperplane at the point $y_0 = [1, 0,\dots, 0]$. With $[z_0, z_1]$ as homogeneous coordinates on this projective line, we find that the index of s at y_0 is equal to $d' - d''$. If τ represents the first Chern class of $H^{d'}\bar{H}^{d''}$, then, according to Theorem 6.1,

$$\int_{CP^n} \tau \wedge \omega^{n-1} = (d' - d'') \int_{CP^{n-1}} \omega^{n-1} = d' - d''.$$

Thus, we may choose τ to be $(d' - d'')\,\omega$, then the total Chern class of $H^{d'}\bar{H}^{d''}$ is $c(H^{d'}\bar{H}^{d''}) = 1 + (d' - d'')[\omega]$, where $[\omega]$ denotes the cohomology class of ω.

The total Chern class of the k-plane bundle E is therefore

$$\begin{aligned} c(E) &= (1 + (d_1' - d_1'')[\omega]) \cdots (1 + (d_k' - d_k'')[\omega]) \\ &= 1 + \cdots + C_{k'}[\omega^{k'}] + \cdots + C_k[\omega^k], \end{aligned}$$

where $C_1 = \sum (d_i' - d_i''),\dots, C_k = \prod (d_i' - d_i'')$. It follows from Theorem 6.2 that

$$\sum \operatorname{Index}\Delta(i) \int_{\Delta(i)} \omega^{n-k'} = \int_{CP^n} C_{k'}\omega^{k'} \wedge \omega^{n-k'} = C_{k'}\,.$$

Hence the formula (8.2) is established.

We would like to point out that, with multiplicities counted, the degeneracy set Δ of r sections $s_1,\dots, s_r$ of the k-plane bundle E in (8.1) is, in general, not homologous to the zero set of a section of the complex vector bundle $\Lambda^r(E)$. For example, if $k = 3$ and $r = 2$, the correct codimension of Δ is $k' = 2$. On the other hand, the fiber dimension of $\Lambda^2(E)$ is 3, and the zero set of a generic section of $\Lambda^2(E)$ should be of codimension 3.

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